Excited States in Spin Chains from Conformal Blocks

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We develop a method of constructing excited states in one dimensional spin chains which are derived from the $SU(2)_1$ Wess-Zumino-Witten Conformal Field Theory (CFT) using a parent Hamiltonian approach. The resulting systems are equivalent to the Haldane-Shastry model. In our ansatz, correlation functions between primary fields correspond to the ground state of the spin system, whereas excited states are obtained by insertion of descendant fields. Our construction is based on the current algebra of the CFT and emphasizes the close relation between the spectrum of the spin system and the underlying CFT. This general structure might imply that the method could be applied to a wider range of model systems.

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I. INTRODUCTION

The description of quantum many body systems is an intrinsically hard problem due to the dimensionality of the Hilbert space, which depends exponentially on the system size. Despite the impressive success of numerical and approximate techniques like Exact Diagonalization[1], Quantum Monte Carlo[2,3], and the Density Matrix Renormalization Group[4], exactly solvable models remain essential in studying the physics of quantum many body systems. An analytical solution is not only indispensable for benchmarking approximate methods, it can also elucidate the general structure of solutions and underlying physical principles. With the experimental advances in cooling and controlling the interactions of atoms, interesting model systems that have a closed theoretical solution may even be engineered and studied in the laboratory.

Given the complexity of a generic quantum many body problem, the direct construction of exactly solvable models appears as an appealing approach. In the past years, $1+1$ dimensional Conformal Field Theory (CFT) has proven to be a powerful tool to construct model Hamiltonians and corresponding ground state wave functions for continuum and lattice quantum Hall models[5–7]. In this approach, a CFT is taken as the starting point to derive a quantum many body system. It can thus contribute to a systematic understanding of systems that admit a description in terms of a scale invariant theory, such as critical systems or systems with topological order that have a gapless edge spectrum. Theories for which such an analysis was carried out include the $SU(2)_k$[8], $SO(n)_1$[9], $U(1)_q$[10], and $SU(n)_1$[11,12] Wess-Zumino-Witten (WZW) theories. Another correspondence between CFT and a class of continuum and lattice quantum systems in two dimensions was studied in Ref. 13. In these models, the ground state exhibits a $z = 2$ Lifshitz scale invariance.

In this paper, we describe a method of constructing excited states of a spin system that is derived from a CFT. Correlation functions of fields in the CFT are interpreted as wave functions of states in the spin system. Following earlier studies[8,14], we consider the $SU(2)_1$ WZW model and define a parent Hamiltonian for the state that corresponds to the correlator of N primary fields. In the case of periodic boundary conditions in one dimension, which we focus on here, the resulting spin system is equivalent to the Haldane-Shastry model[15,16].

Based on solutions to the Calogero-Sutherland model[17–20], excited states of the Haldane-Shastry Hamiltonian were first constructed as polynomials in the particle basis[15,21]. It was realized later[22,23] that these states are the highest weight states of the Yangian algebra, a hidden symmetry of the Haldane-Shastry Hamiltonian. Given the close relation between the Haldane-Shastry model and the $SU(2)_1$ WZW CFT, it was conjectured in Ref. 22 that there should be a correspondence between the states in the CFT and those of the spin chain. Here we show that this is indeed the case. Since our ansatz is based on the $SU(2)$ currents, it is manifestly $SU(2)$ invariant. We relate our results both to the Yangian highest weight states and to the spinon basis of CFT states[24].

Our ansatz mirrors the structure of the CFT, with the ground state corresponding to the CFT vacuum and the excited states corresponding to CFT descendant states. More precisely, we construct the excited states from the ground state by insertion of current operator modes into correlation functions of primary fields. We show that the Hamiltonian of the spin system is block diagonal in these states, with each block corresponding to a fixed number of inserted current operators. This allows us to obtain excited states by successively adding current operators and diagonalizing the blocks. As the Hamiltonian, the Yangian is closed in the subspaces of states with a certain number of current operators. This implies that one can construct representations of the Yangian algebra in these subspaces. We explicitly construct the highest weight states for the analytically obtained eigenstates. Our ansatz for the excited states and its relation to descendant states of the CFT is summarized in Table I.

We perform the diagonalization of the Hamiltonian analytically for up to eight current operators, construct Yangian highest weight states in the obtained eigenstates, and confirm numerically for small system sizes that the

TABLE I. Structure of the states in the CFT and the spin system. The ground state of the spin system is constructed from the product of N primary fields $\Phi_{\mathbf{s}}(\mathbf{z}) = \phi_{s_1}(z_1)\dots\phi_{s_N}(z_N)$ and corresponds to the CFT vacuum $|0\rangle$. Excited states are constructed from the ground state by insertion of current operator modes $J^{a_k}_{-1}\dots J^{a_1}_{-1}$.

	CFT		Spin system
Ground state	$\vert 0\rangle$	$\leftrightarrow$	$\langle 0\vert\Phi_{\mathbf{s}}(\mathbf{z})\vert 0\rangle$
	$\downarrow$		$\downarrow$
Excited states	$(J^{a_k}_{-1}\dots J^{a_1}_{-1})(0)\vert 0\rangle$	$\leftrightarrow$	$\langle 0\vert\Phi_{\mathbf{s}}(\mathbf{z})(J^{a_k}_{-1}\dots J^{a_1}_{-1})(0)\vert 0\rangle$

complete spectrum can be obtained in this way. We show that this construction can be done both for an even and for an odd number of spins in the chain.

This paper is structured as follows: In Section II, we review some properties of the $SU(2)_1$ WZW model and describe the states that form the basis of our construction. We introduce the parent Hamiltonian in Section III and construct excited states, both analytically (Section III C) and numerically (Section III E). We conclude in Section IV.

II. STATES FROM CONFORMAL BLOCKS

In this section, we briefly review some properties of the $SU(2)_1$ WZW model and describe the correspondence between conformal blocks and spin system wave functions.

In addition to the identity, the model has one primary field with scaling dimension $h = 1/4$, the vertex operator $\phi_s(z)$. It can be constructed from the chiral part $\varphi(z)$ of a free, massless boson as

$$\phi_s(z) = e^{i\pi(q-1)(s+1)/2} : e^{is\varphi(z)/\sqrt{2}} : . \qquad (1)$$

Here $s = \pm 1$ corresponds to the two components of the vertex operator, and the colons denote normal ordering. (The holomorphic field $\phi_s(z)$ has an anti-holomorphic counterpart $\bar\phi_s(\bar z)$. In the following we only consider the holomorphic sector.) The value $q \in \{0, 1\}$ corresponds to the two sectors of the CFT: $q = 0$ if the operator $\phi_s(z)$ acts on a state that has an even number of $h = 1/4$ primary fields and $q = 1$ for a state with an odd number, respectively.

In addition to conformal invariance, the $SU(2)_1$ WZW model has an $SU(2)$ symmetry, which is generated by the current operator $J^a(z)$. Its Laurent expansion defines modes J^a_n,

$$J^a(z) = \sum_{n=-\infty}^{\infty} z^{-n-1} J^a_n, \qquad (2)$$

where $a \in \{x, y, z\}$. They satisfy the Kac-Moody algebra[25]

$$\left[J^a_m, J^b_n\right] = i\varepsilon_{abc} J^c_{m+n} + \frac{m}{2}\delta_{ab}\delta_{m+n,0}, \qquad (3)$$

with ε_{abc} being the Levi-Civita symbol and δ_{ab} the Kronecker delta.

A primary field $\phi_s(z)$ transforms covariantly with respect to conformal and $SU(2)$ transformations. The latter is expressed by the operator product expansion (OPE) between the current $J^a(z)$ and $\phi_s(z)$[26],

$$J^a(z)\phi_s(w) \sim -\sum_{s'} \frac{(t^a)_{ss'}}{z-w}\phi_{s'}(w). \qquad (4)$$

Here t^a are the $SU(2)$ spin operators. They are related to the Pauli matrices σ^a by $t^a = \sigma^a/2$.

The modes of the current operator give rise to a tower of descendant states

$$(J^{a_k}_{-1}\dots J^{a_1}_{-1})(0)|0\rangle, \qquad (5)$$

with $|0\rangle$ being the CFT vacuum. States of this form build up the spectrum of the CFT[27]. Note that it suffices to consider states obtained from the $n = -1$ mode of $J^a(z)$. This is so because one can successively rewrite a higher order mode J^a_{-n}, $n > 0$, in terms of the lower order modes J^a_{-n+1} and J^a_{-1} by means of the Kac-Moody algebra (cf. Eq. (3)),

$$J^a_{-n} = \frac{i}{2}\varepsilon_{abc}\left[J^c_{-1}, J^b_{-n+1}\right], n \neq 0. \qquad (6)$$

In this work, we show that the spectrum of the spin systems organizes in the same way in terms of current operators. The excited states we obtain are linear combinations of conformal blocks containing current operator modes $J^{a_k}_{-1}\dots J^{a_1}_{-1}$. We give a summary of these states in Table II and describe the construction of states from conformal blocks in the next subsections.

TABLE II. Summary of the different towers of states obtained by insertion of current operator modes. $\Phi_{\mathbf{s}}(\mathbf{z})$ denotes the product of N primary fields, $\Phi_{\mathbf{s}}(\mathbf{z}) = \phi_{s_1}(z_1)\dots\phi_{s_N}(z_N)$. Using the OPE between $\phi_s(z)$ and $J^a(z)$, the wave functions for these states can be written as the application of k Fourier transformed spin operators to the state without current operators, as explained in Section II B.

	Tower of states	See Eq.
N even	$\langle\Phi_{\mathbf{s}}(\mathbf{z})(J^{a_k}_{-1}\dots J^{a_1}_{-1})(0)\rangle$	(14)
N even	$\langle\phi_{s_\infty}(\infty)\Phi_{\mathbf{s}}(\mathbf{z})(J^{a_k}_{-1}\dots J^{a_1}_{-1}\phi_{s_0})(0)\rangle$	(23)
N odd	$\langle\phi_{s_\infty}(\infty)\Phi_{\mathbf{s}}(\mathbf{z})(J^{a_k}_{-1}\dots J^{a_1}_{-1})(0)\rangle$	(28)
N odd	$\langle\Phi_{\mathbf{s}}(\mathbf{z})(J^{a_k}_{-1}\dots J^{a_1}_{-1}\phi_{s_0})(0)\rangle$	(30)

A. State Obtained from a String of Vertex Operators

We consider an even number of vertex operators $\phi_{s_i}(z_i)$ $(i = 1, \dots, N)$ that each transform under a represen-

tation of $SU(2)$ generated by spin operators t_i^a, as expressed by the OPE of Eq. (4). The key idea is to view the correlation function of vertex operators in the CFT as the wave function of a system of spin 1/2 degrees of freedom on a lattice,

$$|\psi_0\rangle = \sum_{s_1\dots s_N} \psi_0(s_1,\dots,s_N)|s_1,\dots,s_N\rangle, \quad (7)$$

where $\psi_0(s_1,\dots,s_N)$ is given by

$$\psi_0(s_1,\dots,s_N) = \langle \phi_{s_1}(z_1)\dots\phi_{s_N}(z_N)\rangle. \quad (8)$$

Here,

$$\phi_{s_j}(z_j) = e^{\pi i(j-1)(s_j+1)/2} : e^{is_j\varphi(z_j)/\sqrt{2}} :, \quad (9)$$

and $|s_1,\dots,s_N\rangle$ with $s_i = \pm 1$ is the tensor product of eigenstates $|s_i\rangle$ of the z-component of the spin operator t_i^z. The coordinates z_i define the lattice positions in the complex plane and are kept fixed. In contrast to the continuum case, there is no spatial degree of freedom in the basis states $|s_1,\dots,s_N\rangle$ and therefore no integral over the positions in Eq. (7). This is why we use the notation $\psi_0(s_1,\dots s_N)$ without the coordinates z_i for the spin wave function.

The correlation function of N vertex operators is given by[26]

$$\begin{aligned}\psi_0(s_1,\dots,s_N) &= \langle \phi_{s_1}(z_1)\dots\phi_{s_N}(z_N)\rangle \quad (10)\\ &= \delta_{\mathbf{s}}\chi_{\mathbf{s}}\prod_{i<j}^{N}(z_i-z_j)^{s_is_j/2},\end{aligned}$$

where $\delta_{\mathbf{s}}$ is 1 if $\sum_{i=1}^N s_i = 0$ and 0 otherwise. $\chi_{\mathbf{s}}$ is the Marshall sign factor,

$$\chi_{\mathbf{s}} = \prod_{p=1}^{N} e^{i\pi(p-1)(s_p+1)/2}, \quad (11)$$

which ensures that the state ψ_0 is a spin singlet[6,14],

$$T^a\psi_0 = 0, \quad T^a = \sum_{i=1}^{N} t_i^a. \quad (12)$$

Note that the condition $\delta_{\mathbf{s}}$ requires the number of primary fields $\phi_{s_j}(z_j)$ in the correlator to be even.

The positions z_j can, in principle, assume any value on the complex plane. In the following we mostly consider N spins uniformly distributed on the circle,

$$z_j = z^j, \quad \text{with } z = e^{2\pi i/N}. \quad (13)$$

In this case, the state ψ_0 has a momentum of π if $N/2$ is odd and 0 if $N/2$ is even[28], see also Appendix D.

B. States Obtained from Vertex and Current Operators

In analogy to the tower of CFT descendant states $(J_{-1}^{a_k}\dots J_{-1}^{a_1})(0)|0\rangle$, we define a tower of spin states by insertion of modes of the current operator $J^a(z)$ into a correlation function of vertex operators,

$$|\psi_{a_k\dots a_1}\rangle = \sum_{s_1\dots s_N}\psi_{a_k\dots a_1}(s_1,\dots,s_N)|s_1,\dots,s_N\rangle,$$

where $\psi_{a_k\dots a_1}(s_1,\dots,s_N)$ is given by

$$\begin{aligned}&\psi_{a_k\dots a_1}(s_1,\dots,s_N)\\ &\quad= \langle \phi_{s_1}(z_1)\dots\phi_{s_N}(z_N)(J_{-1}^{a_k}\dots J_{-1}^{a_1})(0)\rangle. \quad (14)\end{aligned}$$

It is possible to obtain the states $\psi_{a_k\dots a_1}$ by applying spin operators to ψ_0. To see this, we first note that

$$\langle \Phi_{\mathbf{s}}(\mathbf{z})(J_{-1}^a B)(0)\rangle = \frac{1}{2\pi i}\oint_0 \frac{\mathrm{d}w}{w}\langle \Phi_{\mathbf{s}}(\mathbf{z})J^a(w)B(0)\rangle, \quad (15)$$

with $\Phi_{\mathbf{s}}(\mathbf{z}) \equiv \phi_{s_1}(z_1)\dots\phi_{s_N}(z_N)$ and B being an arbitrary operator. Applying this relation to the definition of $\psi_{a_k\dots a_1}$ we obtain

$$\begin{aligned}&\psi_{a_k\dots a_1}(s_1,\dots,s_N)\\ &= \langle \phi_{s_1}(z_1)\dots\phi_{s_N}(z_N)(J_{-1}^{a_k}\dots J_{-1}^{a_1})(0)\rangle\\ &= \frac{1}{2\pi i}\oint_0\frac{\mathrm{d}w}{w}\langle \Phi_{\mathbf{s}}(\mathbf{z})J^{a_k}(w)(J_{-1}^{a_{k-1}}\dots J_{-1}^{a_1})(0)\rangle\\ &= -\frac{1}{2\pi i}\sum_{j=1}^{N}\oint_{z_j}\frac{\mathrm{d}w}{w}\langle \Phi_{\mathbf{s}}(\mathbf{z})J^{a_k}(w)(J_{-1}^{a_{k-1}}\dots J_{-1}^{a_1})(0)\rangle.\end{aligned} \quad (16)$$

Using the OPE between a current operator and a primary field (cf. Eq. (4)), we get

$$\begin{aligned}&\psi_{a_k\dots a_1}(s_1,\dots,s_N)\\ &= \frac{1}{2\pi i}\sum_{j=1}^{N}\oint_{z_j}\frac{\mathrm{d}w}{w}\frac{t_j^{a_k}}{w-z_j}\langle \Phi_{\mathbf{s}}(\mathbf{z})(J_{-1}^{a_{k-1}}\dots J_{-1}^{a_1})(0)\rangle\\ &= \sum_{j=1}^{N}\frac{t_j^{a_k}}{z_j}\psi_{a_{k-1}\dots a_1}.\end{aligned} \quad (17)$$

Successive application of the same argument results in

$$\psi_{a_k\dots a_1} = \left(\sum_{j_k=1}^{N}\frac{t_{j_k}^{a_k}}{z_{j_k}}\right)\dots\left(\sum_{j_1=1}^{N}\frac{t_{j_1}^{a_1}}{z_{j_1}}\right)\psi_0. \quad (18)$$

If the positions z_j are uniformly distributed on the circle, we can express this result in terms of Fourier transformed spin operators u_l^a,

$$u_l^a \equiv \sum_{j=1}^{N} t_j^a e^{2\pi ijl/N}. \quad (19)$$

With $z = e^{2\pi i/N}$ in Eq. (18), we have

$$\psi_{a_k \ldots a_1} = u_{-1}^{a_k} \ldots u_{-1}^{a_1} \psi_0. \tag{20}$$

Therefore, each additional insertion of J_{-1}^a changes the momentum of the state by $2\pi/N$.

C. States with Additional Spins at Zero and Infinity

We define an additional class of states by inserting two extra vertex operators into the correlator $\langle \phi_{s_1}(z_1) \ldots \phi_{s_N}(z_N) \rangle$, one at $z = 0$ and one at $z = \infty$,

$$|\psi_0^{s_0,s_\infty}\rangle = \sum_{s_1,\ldots,s_N} \psi_0^{s_0,s_\infty}(s_1,\ldots,s_N)|s_1,\ldots,s_N\rangle, \tag{21}$$
$$\psi_0^{s_0,s_\infty}(s_1,\ldots,s_N) = \langle \phi_{s_\infty}(\infty)\phi_{s_1}(z_1)\ldots\phi_{s_N}(z_N)\phi_{s_0}(0)\rangle.$$

In the Riemann sphere picture, the additional spins are added at the south and north pole, respectively, while the N spins at the unit circle are located at the equator (see Fig. 1).

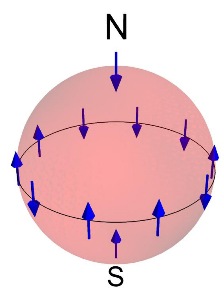

FIG. 1. (Color online) The Riemann sphere with N spins at the equator (unit circle) and two additional spins, one at the north pole ($z = 0$) and one at the south pole ($z = \infty$).

The wave function is

$$\psi_0^{s_0,s_\infty}(s_1,\ldots,s_N) \propto \delta_{\bar{\mathbf{s}}}(-1)^{s_0(1-s_\infty)/2}\chi_{\mathbf{s}} \prod_{n=1}^{N} z_n^{s_0 s_n/2} \times \prod_{n<m}^{N} (z_n - z_m)^{s_n s_m/2}, \tag{22}$$

where $\delta_{\bar{\mathbf{s}}} = 1$ for $s_0 + s_\infty + \sum_{i=1}^N s_i = 0$ and $\delta_{\bar{\mathbf{s}}} = 0$ otherwise.

Note that the extra fields inserted at zero and infinity are primary fields. If we insert additional current operators, we generate descendant states. We thus define a tower of states on top of $\psi_0^{s_0 s_\infty}$,

$$\psi_{a_k\ldots a_1}^{s_0,s_\infty}(s_1,\ldots,s_N) = \langle \phi_{s_\infty}(\infty)\phi_{s_1}(z_1)\ldots\phi_{s_N}(z_N)(J_{-1}^{a_k}\ldots J_{-1}^{a_1}\phi_{s_0})(0)\rangle. \tag{23}$$

This ansatz thus corresponds to the tower of CFT descendant states $(J_{-1}^{a_k}\ldots J_{-1}^{a_1}\phi_s)(0)|0\rangle$.

An argument similar to the one given for $\psi_{a_k\ldots a_1}$ shows that

$$\psi_{a_k\ldots a_1}^{s_0,s_\infty} = \left(\frac{t_\infty^{a_k}}{z_\infty} + \sum_{j_k=1}^{N} \frac{t_{j_k}^{a_k}}{z_{j_k}} \right) \ldots \times \left(\frac{t_\infty^{a_1}}{z_\infty} + \sum_{j_1=1}^{N} \frac{t_{j_1}^{a_1}}{z_{j_1}} \right) \psi_0^{s_0,s_\infty}. \tag{24}$$

Note that the terms $t_\infty^{a_j}/z_\infty$ do not contribute in the limit $z_\infty \to \infty$. On the unit circle, we have

$$\psi_{a_k\ldots a_1}^{s_0,s_\infty} = u_{-1}^{a_k}\ldots u_{-1}^{a_1}\psi_0^{s_0,s_\infty} \tag{25}$$

in terms of Fourier transformed spin operators.

D. Odd Number of Spins

A correlation function of vertex operators $\langle \phi_{s_1}(z_1)\ldots\phi_{s_N}(z_N)\rangle$ is only non-zero if the sum $\sum_{i=1}^N s_i$ vanishes (cf. Eq. (10)). This property, the *charge neutrality condition*, implies that the vertex operators in the correlator need to have a net charge of zero. As a consequence, the number of vertex operators, and therefore the number of spins, needs to be even. We can, however, still consider a model with an odd number of spins at the circle, by adding an extra vertex operator that compensates the excess charge at the circle. Inserting an additional vertex operator at $z = \infty$, we obtain the state

$$\psi_0^{s_\infty}(s_1,\ldots,s_N) = \langle \phi_{s_\infty}(\infty)\phi_{s_1}(z_1)\ldots\phi_{s_N}(z_N)\rangle \propto \delta_{\bar{\mathbf{s}}}(-1)^{(s_\infty+1)/2}\chi_{\mathbf{s}} \prod_{i<j}^{N}(z_i - z_j)^{s_i s_j/2}, \tag{26}$$

where $\delta_{\bar{\mathbf{s}}} = 1$ if $s_\infty + \sum_{i=1}^N s_i = 0$ and $\delta_{\bar{\mathbf{s}}} = 0$ otherwise. The wave function $\psi_0^{s_\infty}$ has spin 1/2,

$$T^a T^a \psi_0^{s_\infty} = \frac{3}{4}\psi_0^{s_\infty}, \quad \text{with } T^a = \sum_{i=1}^{N} t_i^a. \tag{27}$$

This is a consequence of $\psi_0^{s_\infty}$ being a singlet of the total spin including the point $z = \infty$, $(t_\infty^a + T^a)\psi_0^{s_\infty} = 0$. As in the case of an even number of spins, we define a tower of states by insertion of current operators,

$$\psi_{a_k\ldots a_1}^{s_\infty}(s_1,\ldots,s_N) = \langle \phi_{s_\infty}(\infty)\phi_{s_1}(z_1)\ldots\phi_{s_N}(z_N)(J_{-1}^{a_k}\ldots J_{-1}^{a_1})(0)\rangle. \tag{28}$$

We obtain a second class of states by inserting the additional vertex operator at $z=0$ instead of $z=\infty$,

$$\begin{aligned}\psi_0^{s_0}&(s_1,\ldots,s_N)\\&=\langle\phi_{s_1}(z_1)\ldots\phi_{s_N}(z_N)\phi_{s_0}(0)\rangle\\&\propto\delta_{\bar{\mathbf{s}}}\chi_{\mathbf{s}}\prod_{i=1}^N z_i^{s_0s_i/2}\prod_{i<j}^N(z_i-z_j)^{s_is_j/2},\end{aligned}\tag{29}$$

and the corresponding tower

$$\begin{aligned}\psi^{s_0}_{a_k,\ldots,a_1}&(s_1,\ldots,s_N)\\&=\langle\phi_{s_1}(z_1)\ldots\phi_{s_N}(z_N)(J^{a_k}_{-1}\ldots J^{a_1}_{-1}\phi_{s_0})(0)\rangle.\end{aligned}\tag{30}$$

On the unit circle, the two towers of states can be written as

$$\psi^{s_\infty}_{a_k,\ldots,a_1}(s_1,\ldots,s_N)=u^{a_k}_{-1}\ldots u^{a_1}_{-1}\psi_0^{s_\infty}\tag{31}$$

and

$$\psi^{s_0}_{a_k,\ldots,a_1}(s_1,\ldots,s_N)=u^{a_k}_{-1}\ldots u^{a_1}_{-1}\psi_0^{s_0},\tag{32}$$

respectively.

Comparing the wave functions of Eq. (26) and Eq. (29) with the case of an even number of spins (Eq. (10) and (22)), we conclude that $\psi_0^{s_\infty}$ is the analog of ψ_0 and $\psi_0^{s_0}$ of $\psi_0^{s_0s_\infty}$, respectively.

III. PARENT HAMILTONIAN AND SPECTRUM

In this section, we introduce the parent Hamiltonian of ψ_0, discuss its equivalence to the Haldane-Shastry model and construct its spectrum from CFT current operators. Furthermore, we relate our ansatz to the multiplets of the Yangian algebra and the spinon construction of the CFT Hilbert space. From now on we assume a uniform, one-dimensional lattice with periodic boundary conditions,

$$z_j=e^{2\pi ij/N}\tag{33}$$

with $j\in\{1,\ldots,N\}$.

A. Parent Hamiltonian

In earlier work on the Haldane-Shastry model[29], an operator $\mathcal{C}_i^a$ was constructed that annihilates the wave function ψ_0, $\mathcal{C}_i^a\psi_0=0$. This operator was later obtained in a more general setting from the $SU(2)_1$ WZW model using null vectors[8]. In terms of spin operators,

$$\mathcal{C}_i^a=\frac{2}{3}\sum_{j(\neq i)}w_{ij}\left(t_j^a+i\varepsilon_{abc}t_i^bt_j^c\right),\tag{34}$$

with

$$w_{ij}\equiv\frac{z_i+z_j}{z_i-z_j}.\tag{35}$$

We have used the notation $\sum_{i(\neq j)}$ for a sum over all $i\in\{1,\ldots,N\}\setminus\{j\}$.

This allows for the definition of a parent Hamiltonian H of ψ_0[8],

$$H=\frac{1}{4}\sum_{i=1}^N(\mathcal{C}_i^a)^\dagger\mathcal{C}_i^a.\tag{36}$$

Note that H is positive semidefinite and ψ_0 is an eigenstate of H with zero energy.

It is known[8] that H is closely related to the Haldane-Shastry Hamiltonian H_{HS}, if the spins are uniformly distributed on the circle,

$$H_{\mathrm{HS}}=\frac{1}{2}\sum_{i\neq j}\frac{t_i^at_j^a}{\sin^2\left(\frac{(i-j)\pi}{N}\right)}=H+\frac{N+1}{6}T^aT^a+E_0.\tag{37}$$

Here $T^a=\sum_{i=1}^N t^a$ is the total spin and $E_0=-(N^3+5N)/24$ the ground state energy of the Haldane-Shastry Hamiltonian. Note that ψ_0 is annihilated by H and also by T^aT^a in Eq. (37) since it is a singlet. Therefore, ψ_0 is the ground state of the Haldane-Shastry Hamiltonian.

From now on, we will work with the Hamiltonian

$$\mathcal{H}=H+\frac{N+1}{6}T^aT^a,\tag{38}$$

dropping the constant E_0.

B. Block Diagonal Form of the Hamiltonian

We now systematically construct excited states of $\mathcal{H}$ from conformal correlation functions. Specifically, we build the excited states as linear combinations of the states $\psi_{a_k\ldots a_1}$ (cf. Eq. (10) and (18)).

The key to this construction is that the Hamiltonian does not couple states with a fixed number of current operators to states with a different number of current operators, i.e. the Hamiltonian is block-diagonal in this basis.

Therefore, we can diagonalize the Hamiltonian in the subspaces of states with a certain number of current operators. It is not necessary to construct the Hamiltonian in the full Hilbert space of dimension 2^N in order to find eigenstates beyond the ground state. Rather, we obtain eigenstates by successively adding current operators and diagonalizing the blocks.

Let us now show that $\mathcal{H}\psi_{a_k\ldots a_1}$ is a linear combination of states obtained from ψ_0 by insertion of k current operators.

Recall that the insertion of k current operator modes $J_{-1}^{a_j}$ ($j=1,\dots,k$) into the correlation function of vertex operators is equivalent to the successive application of Fourier transformed spin operators $u_{-1}^{a_j}$ to the ground state ψ_0 (cf. Eq. (20)). Therefore, we have computed the commutator between $\mathcal{H}$ and u_{-1}^{a} by an explicit expansion of the Hamiltonian in terms of Fourier modes u_l^a (cf. Section A of the Appendix). The result of this calculation is

$$\left[\mathcal{H},u_{-1}^{a}\right]=(N-1)u_{-1}^{a}+\sum_{i=1}^{N}\frac{3}{2}\frac{\mathcal{C}_i^a}{z_i}+i\varepsilon_{abc}u_{-1}^{b}T^c. \quad (39)$$

From this we can already conclude that the energy of a state with one current operator is $N-1$,

$$\mathcal{H}u_{-1}^{a}\psi_0=\left[\mathcal{H},u_{-1}^{a}\right]\psi_0=(N-1)u_{-1}^{a}\psi_0, \quad (40)$$

since $\mathcal{H}$, $\mathcal{C}_i^a$, and T^c annihilate the ground state ψ_0.

We need to know how $\mathcal{C}_i^a$ acts on $\psi_{a_k\dots a_1}$ to determine the energy of states with more than one current operator. As we show in Section C of the Appendix,

$$\begin{aligned}
&\mathcal{C}_i^a\psi_{a_k\dots a_1}\\
&=\sum_{q=1}^{k}\frac{(K_{a_q}^a)_i}{z_i}\psi_{a_k\dots a_{q+1}a_{q-1}\dots a_1}\\
&\quad+(K_b^a)_iT^b\psi_{a_k\dots a_1}\\
&\quad+2(K_b^a)_i\sum_{q=2}^{k}\sum_{n=0}^{q-1}\frac{i\varepsilon_{ba_qc}}{z_i^{n+1}}\\
&\qquad\times\langle\Phi_{\mathbf{s}}(\mathbf{z})(J_{-1}^{a_k}\dots J_{-1}^{a_{q+1}}J_n^cJ_{-1}^{a_{q-1}}\dots J_{-1}^{a_1})(0)\rangle, \quad (41)
\end{aligned}$$

with

$$\Phi_{\mathbf{s}}(\mathbf{z})=\phi_{s_1}(z_1)\dots\phi_{s_N}(z_N),\quad\text{and} \quad (42)$$

$$(K_b^a)_i=\frac{2}{3}(\delta_{ab}-i\varepsilon_{abc}t_i^c). \quad (43)$$

Combining Eq. (39) and Eq. (41), we obtain

$$\begin{aligned}
&\mathcal{H}\psi_{a_k\dots a_1}\\
&=\sum_{r=1}^{k}u_{-1}^{a_k}\dots u_{-1}^{a_{r+1}}\left[\mathcal{H},u_{-1}^{a_r}\right]u_{-1}^{a_{r-1}}\dots u_{-1}^{a_1}\psi_0\\
&=k(N-1)\psi_{a_k\dots a_1}+\sum_{2\le q<r\le k}\sum_{n=0}^{q-1}F_{a_k\dots a_1}^{qr,n}\\
&\quad+\sum_{1\le q<r\le k}\Big(2\psi_{a_k\dots a_{r+1}a_qa_{r-1}\dots a_{q+1}a_ra_{q-1}\dots a_1}\\
&\qquad-2\delta_{a_ra_q}\psi_{a_k\dots a_{r+1}ca_{r-1}\dots a_{q+1}ca_{q-1}\dots a_1}\\
&\qquad+\psi_{a_k\dots a_{r+1}a_qa_ra_{r-1}\dots a_{q+1}a_{q-1}\dots a_1}\\
&\qquad-\psi_{a_k\dots a_{r+1}a_ra_qa_{r-1}\dots a_{q+1}a_{q-1}\dots a_1}\\
&\qquad+2\delta_{N2}\delta_{a_ra_q}\psi_{a_k\dots a_{r+1}a_{r-1}\dots a_{q+1}a_{q-1}\dots a_1}\Big) \quad (44)
\end{aligned}$$

with

$$\begin{aligned}
&F_{a_k\dots a_1}^{qr,n}\\
&=2\langle\Phi_{\mathbf{s}}(\mathbf{z})(J_{-1}^{a_k}\dots J_{-1}^{a_{r+1}}J_{-n-2}^{a_q}J_{-1}^{a_{r-1}}\dots J_{-1}^{a_{q+1}}\\
&\qquad J_n^{a_r}J_{-1}^{a_{q-1}}\dots J_{-1}^{a_1})(0)\rangle\\
&\quad-2\delta_{a_ra_q}\langle\Phi_{\mathbf{s}}(\mathbf{z})(J_{-1}^{a_k}\dots J_{-1}^{a_{r+1}}J_{-n-2}^{c}J_{-1}^{a_{r-1}}\dots J_{-1}^{a_{q+1}}\\
&\qquad J_n^cJ_{-1}^{a_{q-1}}\dots J_{-1}^{a_1})(0)\rangle\\
&\quad+2N\tilde{\delta}_{n+2}i\varepsilon_{a_ra_qc}\langle\Phi_{\mathbf{s}}(\mathbf{z})(J_{-1}^{a_k}\dots J_{-1}^{a_{r+1}}J_{-1}^{a_{r-1}}\dots J_{-1}^{a_{q+1}}\\
&\qquad J_n^cJ_{-1}^{a_{q-1}}\dots J_{-1}^{a_1}(0))(0)\rangle. \quad (45)
\end{aligned}$$

In the last term, $\tilde{\delta}_{n+2}=1$ if $(n+2)\bmod N=0$ and $\tilde{\delta}_{n+2}=0$ otherwise.

Let us now argue that this expression contains k current operators of order -1. There are two terms that are not yet explicitly written in the desired form, namely the term proportional to δ_{N2} and the term abbreviated by $F_{a_k\dots a_1}^{qr,n}$.

The operators $J_{-n-2}^{a_q}$ (and J_{-n-2}^{c}, respectively), can be written as a linear combination of $n+2$ current operators of order -1 by repeated application of Eq. (6). On the other hand, the operators $J_n^{a_r}$ (and J_n^c, respectively), can be commuted to the right, using the current algebra (cf. Eq. (3))

$$\left[J_n^a,J_{-1}^b\right]=i\varepsilon_{abc}J_{n-1}^c+\frac{1}{2}\delta_{ab}\delta_{n-1,0}. \quad (46)$$

The resulting terms either have n current operators less or vanish because $J_m^a|0\rangle=0$ for $m\ge0$.

The total number of current operators in the first two

terms of $F^{qr,n}_{a_k\dots a_1}$ is therefore

$$k-2+\underbrace{n+2}_{J^{a_q}_{-n-2}}\underbrace{-n}_{J^{a_r}_{n}}=k. \tag{47}$$

If $k \geq N$, there can be a contribution from the third term in $F^{qr,n}_{a_k\dots a_1}$, namely if $n+2=mN$ for $m \in \{1,2,\dots\}$. This term can be written in terms of $k-mN$ current operators. Note, however, that the space of states with k current operators contains the space of states with $k-mN$ current operators. The reason for this is that a current operator J^a_{-1} corresponds to the application of a Fourier transformed spin operator u^a_{-1}, for which $u^a_{-1}=u^a_{-1-mN}$. This argument also applies to the term that is proportional to δ_{N2}.

Thus, the Hamiltonian is block diagonal in the states with a fixed number of current operators.

Note that this observation does not follow from translational invariance only. Translational invariance implies that the Hamiltonian does not mix states with different lattice momenta. Since each current operator in $\psi_{a_k\dots a_1}$ contributes a unit of $2\pi/N$ to the momentum, it follows from translational invariance that $\mathcal{H}\psi_{a_k\dots a_1}$ is a linear combination of states with k mod N current operators. The above considerations moreover show that it is possible to write $\mathcal{H}\psi_{a_k\dots a_1}$ as a linear combination of states with strictly k modes J^a_{-1}. In particular, it is not necessary, to include terms with a higher number of current operators. This allows us to block-diagonalize the Hamiltonian starting with the smaller blocks, i.e. those with a small number of current operators.

C. Eigenstates from Current Operators

We have solved the eigenvalue equation of Eq. (44) for up to eight current operator modes analytically. Since the Hamiltonian is $SU(2)$ invariant, we have decomposed the eigenstates into different spin sectors. The momenta of the states are directly related to the number of current operators, with each current operator changing the momentum by $2\pi/N$. We summarize our results for up to four current operators in Table III.

At level one we find a triplet (spin one), at level two a singlet and a triplet, at level three we find a singlet and two triplets with different energies. A spin two state appears at level 4 as the symmetric traceless part of a state with two non-contracted $SU(2)$-indices.

Note that the number of eigenstates is smaller than the number of possible combinations we can build with k current operators. The reason is that some CFT states $J^{a_k}_{-1}\dots J^{a_1}_{-1}|0\rangle$ are null, such that the norm of the corresponding spin state vanishes. At level three, for example, there is the null state $\sum_b(3\psi^{bab}_0+3\psi^{bba}_0-2\psi^{abb}_0)$.

The number of states that we find with a certain number of current operators is in agreement with the characters of the $SU(2)_1$ algebra[8]: At level 0, 1, 2, 3, and 4 we find 1, 3, 4, 7, and 13 states, respectively. The size of the matrices that need to be diagonalized at a given level in current operators thus corresponds to the characters of $SU(2)_1$.

When considering the action of the Hamiltonian on states with k current operators, there are two types of terms that depend on the number of spins N, cf Eq. (44). The first one, $k(N-1)\psi_{a_k\dots a_1}$, is already diagonal. All other terms only appear if $k\geq N$. These terms are strictly upper triangular in the sense that they can be written in terms of $k-mN$ current operators with $m>0$. This upper triangular structure is preserved by a diagonalization of all other terms. Therefore, only the diagonal term $k(N-1)\psi_{a_k\dots a_1}$ contributes to the N-dependence of the energies.

We thus arrive at an N-independent representation of the energies by subtracting the contribution $k(N-1)$ from the energies,

$$\tilde{E}=\begin{cases}\frac{E-(N-1)k}{k}, & k>0,\\ 0, & k=0.\end{cases} \tag{48}$$

(We have also rescaled the energies by $1/k$ for convenience.)

The shifted and rescaled energies $\tilde{E}$ as well as the spin content of the corresponding eigenspaces are plotted for up to eight current operators in Fig. 2.

Note that for a given N, some of these energies do not occur because the corresponding states have zero norm. These null states appear dependent on N and in addition to null states identified at the CFT level. The occurrence of additional null states reflects the fact that the CFT Hilbert space is infinite, while the spin system's Hilbert space is finite.

A necessary condition for a particular state to be non-null is given by $E\geq 0$. The region of energies, where $E\geq 0$ is satisfied is indicated by the bands in Fig. 2. Depending on N, we find certain states that violate this condition and are therefore null. These states then lead to further null states at higher levels through the application of additional current operators. We find that if a state corresponding to a certain energy is null for a number of spins N', it is also null for N spins with $N\leq N'$. For each energy level, the highest N' that we found exploiting the energy condition $E\geq 0$ is given in Fig. 2 below the spin content.

Let us give an example for the notation used in Fig. 2. At $k=7$ and $\tilde{E}=-6$ we find two spin 1 states and one spin 2 state. One of the spin 1 states is null for all $N\leq 6$, the other spin 1 state and the spin 2 state are null for all $N\leq 8$. This multiplet is shown in Fig. 2 as

$$\boxed{\begin{array}{ccccc}-6 & & & & \\ 1 & \oplus & 1 & \oplus & 2\\ 6 & & 8 & & 8\end{array}}. \tag{49}$$

Note that the violation of the energy condition $E\geq 0$ is sufficient for a state to be null. A complete separation

TABLE III. Eigenstates of $\mathcal{H}$ in terms of states obtained by insertion of k current operator modes for $k \leq 4$. The momentum of the ground state is $p_0 = \pi$ if $N/2$ is odd and $p_0 = 0$ if $N/2$ is even.

k	State	Null for	Energy	Spin	Momentum	Number of states
0	$\varphi^{(0)} = \psi_0$		0	0	p_0	1
1	$\varphi_a^{(1)} = \psi_a$		$N-1$	1	$p_0 - 2\pi/N$	3
2	$\varphi^{(2)} = \sum_c \psi_{cc} - 3\delta_{N2}\psi_0$	$N \leq 2$	$2(N-3)$	0	$p_0 - 4\pi/N$	1
2	$\varphi_a^{(3)} = \sum_{cd} \varepsilon_{acd}\psi_{cd}$	$N \leq 2$	$2(N-3)$	1	$p_0 - 4\pi/N$	3
3	$\varphi^{(4)} = \sum_{cde} \varepsilon_{cde}\psi_{cde}$	$N \leq 4$	$3(N-5)$	0	$p_0 - 6\pi/N$	1
3	$\varphi_a^{(5)} = \sum_c (2\psi_{acc} - 3\psi_{cac} + \psi_{cca}) - 4\delta_{N2}\psi_a$	$N \leq 4$	$3(N-5)$	1	$p_0 - 6\pi/N$	3
3	$\varphi_a^{(6)} = \sum_c (\psi_{cca} - \psi_{acc}) + 2\delta_{N2}\psi_a$	$N \leq 2$	$3(N-3)$	1	$p_0 - 6\pi/N$	3
4	$\varphi^{(7)} = \sum_{cd} (5\psi_{cdcd} - 3\psi_{cddc} - 2\psi_{ccdd}) + 16\delta_{N4}\psi_0 + 12\delta_{N2}\psi_0$	$N \leq 6$	$4(N-7)$	0	$p_0 - 8\pi/N$	1
4	$\varphi_a^{(8)} = \sum_{cde} (4\varepsilon_{acd}\psi_{cdee} - 3\varepsilon_{acd}\psi_{cede})$	$N \leq 6$	$4(N-7)$	1	$p_0 - 8\pi/N$	3
4	$\varphi^{(9)} = \sum_{cd} (\psi_{cddc} - \psi_{ccdd}) - 12\delta_{N4}\psi_0 + 6\delta_{N2}\psi_0$	$N \leq 4$	$4N-18$	0	$p_0 - 8\pi/N$	1
4	$\varphi_a^{(10)} = \sum_{cde} (\varepsilon_{acd}\psi_{cdee} + 3\varepsilon_{acd}\psi_{cede})$	$N \leq 4$	$4N-18$	1	$p_0 - 8\pi/N$	3
4	$\varphi_{ab}^{(11)} = \sum_c \left(\frac{1}{2}(\psi_{abcc} + \psi_{bacc}) - \frac{1}{3}\delta_{ab}\sum_d \psi_{ddcc}\right)$	$N \leq 2$	$4N-10$	2	$p_0 - 8\pi/N$	5

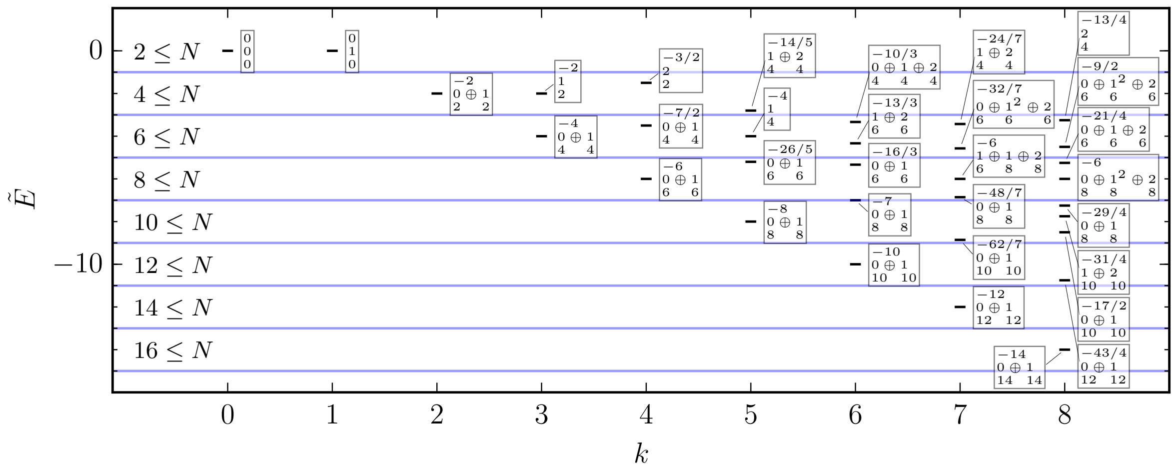

FIG. 2. (Color online) Analytically calculated energy levels and their spin content as a function of the number of current operators k. $\tilde{E}$ is defined as $\tilde{E} = (E - (N-1)k)/k$ for $k > 0$ and $\tilde{E} = E = 0$ for $k = 0$. The three rows in the boxes next to the energy level correspond to the calculated values of (1) the energy $\tilde{E}$ (2) the spin content (3) a value for the number of spins N' so that the corresponding state is null for all systems with $N \leq N'$ spins. The inequalities indicate for which N the condition $E \geq 0$ is satisfied. For a given number of spins N, all eigenvalues that lie in bands where the inequality is not satisfied correspond to null states. By applying additional current operators to these null states, we could identify further null states that do not violate the energy condition.

of null states requires the computation of inner products between our ansatz states at the level of the spin system. As shown in Ref. 8, the spin correlation functions in ψ_0 can be computed by solving linear algebraic equations. Using these correlation functions, one could compute the norms of the states $\psi_{a_k \ldots a_1}$ numerically, even for large system sizes.

D. Construction of Yangian Highest Weight States and Comparison to Spinon Basis

The spectrum of the Haldane-Shastry model has previously been constructed by exploiting a hidden symmetry of the Hamiltonian, which is generated by the rapidity operator Λ^a[22],

$$\Lambda^a = \frac{i}{2}\sum_{i \neq j} w_{ij}\varepsilon_{abc} t_i^b t_j^c = \frac{i}{2}\sum_{i \neq j} w_{ij}(\vec{t}_i \times \vec{t}_j)^a. \tag{50}$$

Using $\sum_{j(\neq i)} w_{ij} = 0$, it follows that

$$\Lambda^a = \frac{3}{4}\sum_{i=1}^{N} \mathcal{C}_i^a. \quad (51)$$

The rapidity Λ^a and the total spin T^a form a basis of the Yangian algebra. They both commute with the Haldane-Shastry Hamiltonian, but the rapidity Λ^a does not commute with the total spin operator T^aT^a. This is the reason for the degenerate energy levels formed by multiplets with different total spin[22].

The key to the construction of eigenstates in this approach is the notion of a Yangian highest weight state h, which is annihilated by $\Lambda^+ = \Lambda^x + i\Lambda^y$ and $T^+ = T^x + iT^y$. A multiplet of states with the same energy is then given by application of powers of $\Lambda^- = \Lambda^x - i\Lambda^y$ to h.

In order to relate our method to this approach, we have computed the action of Λ^a on $\psi_{a_k\dots a_1}$ using the decoupling equation derived in section C of the Appendix. We find

$$\begin{aligned} 2\Lambda^a\psi_{a_k\dots a_1} =& \sum_{q=1}^{k}(N-1)i\varepsilon_{aa_qc}\psi_{a_k\dots a_{q+1}ca_{q-1}\dots a_1} \\ &+ \sum_{q=1}^{k} i\varepsilon_{a_qac}\psi_{ca_k\dots a_{q+1}a_{q-1}\dots a_1} \\ &+ \sum_{q=2}^{k}\sum_{n=0}^{q-1} G^{q,n}_{a_k\dots a_q}, \end{aligned} \quad (52)$$

with

$$\begin{aligned} &G^{q,n}_{a_k\dots a_q} = \\ &2\langle\Phi_{\mathbf{s}}(\mathbf{z})(J^{a_q}_{-n-1}J^{a_k}_{-1}\dots J^{a_{q+1}}_{-1}J^{a}_{n}J^{a_{q-1}}_{-1}\dots J^{a_1}_{-1})(0)\rangle \\ &- 2\delta_{a_qa}\langle\Phi_{\mathbf{s}}(\mathbf{z})(J^{c}_{-n-1}J^{a_k}_{-1}\dots J^{a_{q+1}}_{-1}J^{c}_{n}J^{a_{q-1}}_{-1}\dots J^{a_1}_{-1})(0)\rangle \\ &+ 2N\tilde{\delta}_{n+1}i\varepsilon_{aa_qc}\langle\Phi_{\mathbf{s}}(\mathbf{z})(J^{a_k}_{-1}\dots J^{a_{q+1}}_{-1} \\ &\qquad J^{c}_{n}J^{a_{q-1}}_{-1}\dots J^{a_1}_{-1})(0)\rangle. \end{aligned} \quad (53)$$

Furthermore, we have (cf. Eq. (C6) in Appendix C)

$$T^a\psi_{a_k\dots a_1} = i\sum_{q=1}^{N}\varepsilon_{aa_qc}\psi_{a_k\dots a_{q+1}ca_{q-1}\dots q_1}. \quad (54)$$

This shows that the Yangian, like the Hamiltonian, leaves the subspaces of states with a fixed number of current operators invariant. It is thus possible to write the highest weight states of the Yangian algebra in terms of the states $\psi_{a_k\dots a_1}$. We have computed the highest weight states for up to four current operators and expanded the result in the eigenstates listed in Table III. These states thus correspond to the eigenstates constructed by Haldane in Ref. 21, which have a polynomial form in the particle basis and were identified as the highest weight states of the Yangian algebra in Ref. 22. We summarize our results in Table IV.

TABLE IV. Highest weight states of the Yangian algebra in terms of the eigenstates of the Hamiltonian given in Table III.

k	State	Energy	Spin
0	φ^0	0	0
1	$\varphi^{(1)}_x + i\varphi^{(1)}_y$	$N-1$	1
2	$\varphi^{(3)}_x + i\varphi^{(3)}_y$	$2(N-3)$	1
3	$\varphi^{(5)}_x + i\varphi^{(5)}_y$	$3(N-5)$	1
3	$\varphi^{(6)}_x + i\varphi^{(6)}_y$	$3(N-3)$	1
4	$\varphi^{(8)}_x + i\varphi^{(8)}_y$	$4(N-7)$	1
4	$\varphi^{(10)}_x + i\varphi^{(10)}_y$	$4N-18$	1
4	$\varphi^{(11)}_{zz} + 2\varphi^{(11)}_{xx} + 2i\varphi^{(11)}_{xy}$	$4N-10$	2

Note that our ansatz is manifestly $SU(2)$ invariant, whereas the construction in terms of highest weight states is not. The states $\psi_{a_k\dots a_1}$ have a simple form in the sense that they are created from the ground state by successive application of Fourier transformed spin operators (cf. Eq. (20)).

We have compared the highest weight states of the spin system to the highest weight states of the Yangian algebra in the CFT, which is spanned by[22] J^a_0 and

$$Q^a = \frac{i}{2}\sum_{m=1}^{\infty}\varepsilon_{abc}J^b_{-m}J^c_m. \quad (55)$$

We find that the highest weight states that we computed in the spin system are precisely the highest weight states of J^a_0 and Q^a, when seen as states of the CFT under the correspondence of Table I.

Finally, let us relate our ansatz to the spinon basis of the CFT Hilbert space[24]. In this approach, states are not constructed by application of modes of the affine current J^a_{-n} to the CFT vacuum, but by application of modes $\phi_{s,-m}$ of the primary field $\phi_s(z)$. These modes are defined with respect to the Laurent expansion[24]

$$\phi_s(z) = \sum_m z^{m+\frac{q}{2}}\phi_{s,-m-\frac{1}{4}-\frac{q}{2}}. \quad (56)$$

Here $q \in \{0,1\}$ corresponds to the sector of the CFT Hilbert space on which $\phi_s(z)$ is acting: $q=0$ for a state built from an even number of spinon modes and $q=1$ for a state with an odd number of spinon modes. The authors of Ref. 24 derived generalized commutation relations for the modes of $J^a(z)$ and $\phi_s(z)$, which can be used to rewrite states built from current operator modes in terms of spinon modes $\phi_{s,-m}$. At level $k=1$, for example, the state $\langle\phi_{s_1}(z_1)\dots\phi_{s_N}(z_N)J^a_{-1}(0)\rangle$ corresponds to

$$\sum_{ss'}(t^a)_{ss'}\langle\phi_{s_1}(z_1)\dots\phi_{s_N}(z_N)\left(\phi_{-s,-\frac{3}{4}}\phi_{s',-\frac{1}{4}}\right)(0)\rangle. \quad (57)$$

A general state with k current operators of order -1 will be a linear combination of terms with spinon modes $\phi_{\alpha_l,-m_l}\dots\phi_{\alpha_1,-m_1}$ with $k=\sum_{i=1}^{l} m_i$.

E. Complete Spectrum

In the previous subsections, we have analytically diagonalized the Hamiltonian in the states $\psi_{a_k\ldots a_1}$ for $k \leq 8$ and N even. For a given k, the size of the matrices that need to be diagonalized and therefore the complexity of the analytical calculation does not depend on the number of spins N. However, it becomes increasingly difficult for larger k. In order to test if our method yields all excited states or just a subset thereof, we have thus constructed the states $\psi_{a_k\ldots a_1}$ numerically for small N and performed a numerical diagonalization of the Hamiltonian in that basis. Our numerical calculations confirm that the complete spectrum is indeed obtained from our ansatz states.

In an analogous way to ψ_0, we studied the tower of states obtained from $\psi_0^{s_0,s_\infty}$ by insertion of current operators (cf. Section II C). We find numerically that the complete Hilbert space is generated, starting from $\psi_0^{s_0 s_\infty}$ with $s_0, s_\infty \in \{-1, 1\}$ and successively inserting current operator modes. As for ψ_0, the Hamiltonian is block diagonal in the states with a fixed number of current operators.

Furthermore, we constructed the spectrum numerically for the case of an odd number of spins (cf. Section II D). We find that the Hamiltonian is block diagonal in the states $\psi^{s_0}_{a_k\ldots a_1}$ and $\psi^{s_\infty}_{a_k\ldots a_1}$ and that the complete Hilbert space can be constructed from states of this form.

The numerically obtained spectra are shown for $N = 7$ and $N = 8$ spins in Fig. 3.

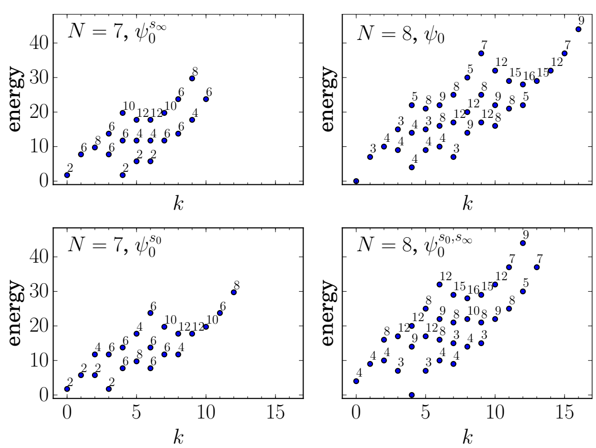

FIG. 3. Numerically calculated spectra for the odd-N spin chain ($N = 7$, left panels) and the even-N spin chain ($N = 8$, right panels). The upper panels show data obtained by applying current operators to the states $\psi_0^{s_\infty}$ and ψ_0, respectively. In the lower panels, the current operators were applied to the states $\psi_0^{s_0}$ and $\psi_0^{s_0 s_\infty}$, respectively. The horizontal axis shows the number of current operators k that are present in the corresponding states. The numbers above the levels indicate their degeneracy. Notice that the spectrum is the same in the upper and lower plots, but a given state does not necessarily appear at the same number of current operators.

We observe a tendency that states with a higher number of current operators have a larger energy. This means that by inserting few current operators, we get access to low-lying excited states. Note, however, that this relation does not hold in a strict sense; it may happen that, by inserting additional current operators, we obtain states with a lower energy.

We observe that the spectra shown in the upper and in the lower panels of Fig. 3 are the same. This means that for both N even and N odd, the complete spectrum is obtained for either of the two classes of states.

The top right panel of Fig. 3 corresponds to our analytical results shown in Fig. 2. Note that not all states of Fig. 2 appear in the numerically calculated spectrum, because certain states of Fig. 2 are null for $N = 8$. This is the case for the all states with $E < 0$, which appear in the bands below $\tilde{E} = -7$ in Fig. 2. The remaining null states have the values

k	$\tilde{E}$	E	Spin
6	-7	0	$0 \oplus 1$
7	$-\frac{48}{7}$	1	$0 \oplus 1$
7	-6	7	$1 \oplus 2$
8	-6	8	$0 \oplus 1^2 \oplus 2$

and can be obtained from null states violating the energy condition by the insertion of additional current operators.

We have numerically computed the maximal number of current operator insertions of order -1 needed to generate the complete spectrum as a function of N. The results are shown for the tower of states built on ψ_0 (N even) in Fig. 4. In this case, our numerical data suggests that $(N/2)^2$ current operators are sufficient to obtain the complete spectrum.

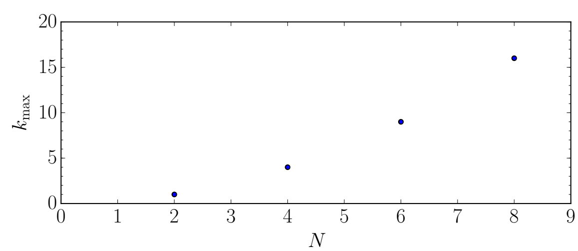

FIG. 4. Maximal number of current operator insertions $k_{\max}$ of order -1 needed to obtain the spectrum of the Hamiltonian dependent on the number of spins N. The data is consistent with $k_{max} = (N/2)^2$.

Finally, let us comment on how the first excited states of the Haldane-Shastry model for N even are related to the states $\psi_{a_k\ldots a_1}$. Except for $N = 2$, these cannot be given by the states at level one, ψ_{a_1}, which have $E = N-1$. The reason is that the states $\psi_0^{s_0,s_\infty}$ are eigenstates with a lower energy of $E = N/2$. This follows from

$$\mathcal{C}_i^a \psi_0^{s_0,s_\infty} = (K_b^a)_i (t_\infty^b - t_0^b) \psi_0^{s_0,s_\infty} \tag{58}$$

and

$$(\mathcal{C}_i^a)^\dagger = \mathcal{C}_i^a - \frac{4}{3}\sum_{j(\neq i)} w_{ij} t_j^a. \tag{59}$$

Note that the latter equality is only valid on the circle with $z_j = e^{2\pi i j/N}$. Our numerical calculations indicate that the states $\psi_0^{s_0,s_\infty}$ are indeed the first excited states. Furthermore, we find a multiplet with spin content $0 \oplus 1$ and energy $N/2$ at $k = N/2$ among the states constructed from $\psi_{a_k \ldots a_1}$, both in the numerical and in the analytical calculations. These multiplets appear at $\tilde{E} = -2k + 2$ in the analytically computed spectrum shown in Fig. 2. Noting that the states $\psi_0^{s_0,s_\infty}$ can be decomposed into a singlet and a triplet, this suggests that the first excited states are given by linear combinations of states $\psi_{a_k \ldots a_1}$ with $k = N/2$ current operator modes.

IV. CONCLUSION

We studied a model of a quantum spin chain that is constructed from the $SU(2)_1$ WZW model and is equivalent to the Haldane-Shastry model. We have shown that it is possible to construct excited states of the spin system from CFT current operators in a way, which directly reflects the structure of the CFT spectrum.

In the approach pursued in this work, correlation functions of operators in the CFT are interpreted as wave functions of spin states. Based on earlier work, where the correlation function of *primary* operators were interpreted as the ground state wave function, we provided a method of constructing excited states by inserting *descendant* fields.

In the case of an even number of spins N, we have diagonalized the Hamiltonian analytically for $k \leq 8$ inserted current operators. Depending on N, we identified certain states that are null but correspond to non-null CFT states. These additional null states occur due to the finite size of the spin system's Hilbert space. Our method of detecting them is based on a sufficient criterion for a state to be null. In order to test if a given state is not null, one could use the known algebraic equations for spin correlation functions[8] in ψ_0 to compute the needed inner products numerically, even for large system sizes.

We have given numerical evidence that this method yields the complete spectrum for a given number of spins N, independent of which primary state we build the tower states from. Furthermore, we have shown numerically that a similar construction can be made for an odd number of spins.

Our manifestly $SU(2)$ invariant ansatz is compatible with the construction of eigenstates of the Haldane-Shastry model as multiplets of the Yangian algebra in the sense that the Yangian operator does not change the number of current operators. This allowed us to explicitly relate the excited states constructed from current operators to the highest weight states of the Yangian operator. Furthermore, we have argued that our ansatz wave functions with k current operator modes of order -1 can be rewritten in terms of a wave function with spinon modes whose mode numbers sum up to $-k$.

In the case of the $SU(2)_1$ WZW model, which we studied here, the resulting spin system is equivalent to the Haldane-Shastry model. Thus our method provides an alternative way of constructing the excited states of the Haldane-Shastry model, which emphasizes its close relation to the underlying CFT. We expect that this method could be generalized to the $SU(n)_1$ WZW model, which is related to the $SU(n)$ Haldane-Shastry spin chain[30]. Another generalization of our construction could be possible within the Laughlin lattice models with filling fraction $1/q$[10]. Even though these systems do not have a Yangian symmetry for $q > 2$, part of the spectrum is described by integer eigenvalues. We have carried out exemplary numerical calculations for $q = 3$ which indicate that at least some of the excited states can be obtained by insertion of current operators.

It is crucial for the construction used above that the system is one-dimensional with periodic boundary conditions. This can be seen, for example, noting that the ansatz $\psi_{a_k \ldots a_1}$ is a decomposition into momentum space eigenstates. However, the states obtained by insertion of current operators might still describe low energy eigenstates of an interesting, two-dimensional system with a different Hamiltonian. We plan to investigate the properties of these states in two dimensions in future work.

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Appendix A: Commutator Between $\mathcal{H}$ and u_{-1}^a

In this section we derive an expression for the commutator $\left[\mathcal{H}, u_{-1}^a\right]$. This commutator was used in Section III B to determine the action of $\mathcal{H}$ on a state $\psi_{a_k \ldots a_1}$. The explicit form of the Hamiltonian in terms of spin operators is[8]

$$\begin{aligned}\mathcal{H} = &-\frac{1}{6}\sum_{k,j}\sum_{i\neq(k,j)} w_{ij} w_{ik} t_j^b t_k^b \\ &- \frac{1}{6}\sum_{i\neq j} w_{ij}^2 t_i^b t_j^b + \frac{1}{6}(N+1) T^b T^b.\end{aligned} \tag{A1}$$

We use the Fourier transforms

$$\sum_{i\neq j}\frac{w_{ij}}{z_i^k z_j^l}=\begin{cases}0, & \text{if } k=0\\ (2k-N)\tilde{\delta}_{k+l}N, & \text{if } k=1,2,\ldots,N-1\end{cases}$$

$$\sum_{i\neq j}\frac{w_{ij}^2}{z_i^k z_j^l}=\begin{cases}(N-N^2/3-2/3)\tilde{\delta}_l N,\\ \quad \text{if } k=0\\ (N^2/6-2(k-N/2)^2-2/3)\tilde{\delta}_{k+l}N,\\ \quad \text{if } k=1,2,\ldots,N-1.\end{cases} \tag{A2}$$

to rewrite $\mathcal{H}$ in terms of u_k^a. Here $\tilde{\delta}_m=1$ if $m \bmod N=0$ and $\tilde{\delta}_m=0$ otherwise. A derivation of these formulas is given in Section B of the Appendix, see also Ref. 28, where these Fourier sums were evaluated using contour integrals.

We find

$$\mathcal{H}=\sum_{k=0}^{N-1}\frac{1+2N^2-9Nk+9k^2}{9N}u_{-k}^b u_k^b. \tag{A3}$$

Starting from this expansion, we next compute the commutator between $\mathcal{H}$ and u_{-1}^a. Using

$$\left[u_m^a,u_n^b\right]=i\varepsilon_{abc}u_{n+m}^c, \tag{A4}$$

which directly follows from the commutator algebra of the spin operators t_i^a, we obtain

$$\left[\mathcal{H},u_{-1}^a\right]=2i\varepsilon_{abc}u_{-1}^b T^c-\frac{2i}{N}\varepsilon_{abc}\sum_{m=1}^{N-1}m u_{-m}^b u_{m-1}^c + (3-N)u_{-1}^a. \tag{A5}$$

It is possible to rewrite the sum over Fourier transformed spin operators in terms of the operator $\mathcal{C}_i^a$, which was defined in Eq. (34). To this end, we computed the Fourier expansion of $\mathcal{C}_i^a$. The result is

$$\frac{3}{2}\sum_{j=1}^{N}\frac{\mathcal{C}_j^a}{z_j^p}=i\varepsilon_{abc}u_{-p}^b T^c+(2p+2-2N)u_{-p}^a -\frac{2i}{N}\varepsilon_{abc}\sum_{m=1}^{N-1}m u_{-m}^b u_{m-p}^c, \tag{A6}$$

for $p\in\{1,\ldots,N-1\}$. Note that for $p=1$ we find the same sum over Fourier transformed spin operators that occurs in the above expansion of the Hamiltonian (Eq. (A5)). Inserting the $p=1$ Fourier mode of C_i^a, we arrive at

$$\left[\mathcal{H},u_{-1}^a\right]=(N-1)u_{-1}^a+i\varepsilon_{abc}u_{-1}^b T^c+\frac{3}{2}\sum_{i=1}^{N}\frac{\mathcal{C}_i^a}{z_i}. \tag{A7}$$

Appendix B: Fourier Transform of w-Functions

The Fourier transforms of $w_{ij}=(z_i+z_j)/(z_i-z_j)$ and w_{ij}^2 can both be reduced to one sum by translational invariance,

$$\sum_{i\neq j}\frac{w_{ij}}{z_i^k z_j^l}=N\tilde{\delta}_{k+l}\sum_{i(\neq N)}\frac{w_{iN}}{z_i^k}, \tag{B1}$$

$$\sum_{i\neq j}\frac{w_{ij}^2}{z_i^k z_j^l}=N\tilde{\delta}_{k+l}\sum_{i(\neq N)}\frac{w_{iN}^2}{z_i^k}. \tag{B2}$$

In order to evaluate the remaining sums, it is useful to compute the Fourier sums

$$\sum_{k=0}^{N-1}k^n z_k^j \tag{B3}$$

for $n=1$ and $n=2$. For $j=0$, we have

$$\sum_{k=0}^{N-1}k=\frac{N(N-1)}{2}, \tag{B4}$$

$$\sum_{k=0}^{N-1}k^2=\frac{N(N-1)(2N-1)}{6}. \tag{B5}$$

For $j=1,2,\ldots,N-1$, we use the generating function

$$f(\omega)=\sum_{k=0}^{N-1}e^{i\omega k}=\frac{1-e^{i\omega N}}{1-e^{i\omega}}, \tag{B6}$$

$$\sum_{k=0}^{N-1}k^n z_k^j=\left(\frac{\mathrm{d}}{i\mathrm{d}\omega}\right)^n f(\omega)\Big|_{\omega=2\pi j/N}.$$

Taking the first and the second derivative, we find

$$\sum_{k=0}^{N-1}k z_k^j=\frac{N}{2}(w_{jN}-1),$$

$$\sum_{k=0}^{N-1}k^2 z_k^j=\frac{N}{2}(1-N+Nw_{jN}-w_{jN}^2). \tag{B7}$$

Taking the inverse Fourier transforms of these equations and solving for $\sum_{j(\neq N)}w_{jN}/z_j^k$ and $\sum_{j(\neq N)}w_{jN}^2/z_j^k$, we arrive at Eq. (A2).

Appendix C: Decoupling Equation for States with Current Operators

The starting point for the derivation of the action of $\mathcal{C}_i^a$ on a state

$$\psi_{a_k\ldots a_1}(s_1,\ldots,s_N) =\langle\phi_{s_1}(z_1)\ldots\phi_{s_N}(z_N)(J_{-1}^{a_k}\ldots J_{-1}^{a_1})(0)\rangle \tag{C1}$$

is the null operator[8]

$$(K_b^a)_i(J_{-1}^b\varphi_{s_i})(z_i), \quad \text{with} \qquad (K_b^a)_i=\frac{2}{3}(\delta_{ab}-i\varepsilon_{abc}t_i^c). \tag{C2}$$

When inserting the null operator into a correlation function of primary fields, the resulting correlator vanishes. We therefore have

$$
\begin{aligned}
0 &= (K^a_b)_i \langle \phi_{s_1}(z_1)\dots(J^b_{-1}\phi_{s_i})(z_i)\dots\phi_{s_N}(z_N)(J^{a_k}_{-1}J^{a_{k-1}}_{-1}\dots J^{a_1}_{-1})(0)\rangle \\
&= (K^a_b)_i \oint_0 \frac{\mathrm{d}w_k}{2\pi i w_k}\cdots\oint_0 \frac{\mathrm{d}w_1}{2\pi i w_1}\langle \phi_{s_1}(z_1)\dots(J^b_{-1}\phi_{s_i})(z_i)\dots\phi_{s_N}(z_N)J^{a_k}(w_k)\dots J^{a_1}(w_1)\rangle \\
&= (K^a_b)_i \oint_0 \frac{\mathrm{d}w_k}{2\pi i w_k}\cdots\oint_0 \frac{\mathrm{d}w_1}{2\pi i w_1}\oint_{z_i}\frac{\mathrm{d}z}{2\pi i(z-z_i)}\langle \phi_{s_1}(z_1)\dots J^b(z)\phi_{s_i}(z_i)\dots\phi_{s_N}(z_N)J^{a_k}(w_k)\dots J^{a_1}(w_1)\rangle \\
&= -(K^a_b)_i \sum_{j(\neq i)}\oint_0 \frac{\mathrm{d}w_k}{2\pi i w_k}\cdots\oint_0 \frac{\mathrm{d}w_1}{2\pi i w_1}\oint_{z_j}\frac{\mathrm{d}z}{2\pi i(z-z_i)}\langle \phi_{s_1}(z_1)\dots J^b(z)\phi_{s_i}(z_i)\dots\phi_{s_N}(z_N)J^{a_k}(w_k)\dots J^{a_1}(w_1)\rangle \\
&\quad -(K^a_b)_i \sum_{q=1}^{k}\oint_0 \frac{\mathrm{d}w_k}{2\pi i w_k}\cdots\oint_0 \frac{\mathrm{d}w_1}{2\pi i w_1}\oint_{w_q}\frac{\mathrm{d}z}{2\pi i(z-z_i)}\langle \phi_{s_1}(z_1)\dots J^b(z)\phi_{s_i}(z_i)\dots\phi_{s_N}(z_N)J^{a_k}(w_k)\dots J^{a_1}(w_1)\rangle. \quad \text{(C3)}
\end{aligned}
$$

Inserting the OPE between a primary field and a current operator (cf. Eq. (4)) and the OPE between two current operators[25],

$$
J^a(z)J^b(w) \sim \frac{\delta_{ab}}{2(z-w)^2} + i\varepsilon_{abc}\frac{J^c(w)}{z-w}, \quad \text{(C4)}
$$

and writing $\Phi_{\mathbf{s}}(\mathbf{z}) = \phi_{s_1}(z_1)\dots\phi_{s_N}(z_N)$, we get

$$
\begin{aligned}
0 &= (K^a_b)_i \sum_{j(\neq i)}\oint_0 \frac{\mathrm{d}w_k}{2\pi i w_k}\cdots\oint_0 \frac{\mathrm{d}w_1}{2\pi i w_1}\oint_{z_j}\frac{\mathrm{d}z}{2\pi i(z-z_i)}\frac{t^b_j}{z-z_j}\langle \Phi_{\mathbf{s}}(\mathbf{z})J^{a_k}(w_k)\dots J^{a_1}(w_1)\rangle \\
&\quad -(K^a_b)_i \sum_{q=1}^{k}\oint_0 \frac{\mathrm{d}w_k}{2\pi i w_k}\cdots\oint_0 \frac{\mathrm{d}w_1}{2\pi i w_1} \\
&\quad \times \oint_{w_q}\frac{\mathrm{d}z}{2\pi i(z-z_i)}\frac{\delta_{ba_q}}{2(z-w_q)^2}\langle \Phi_{\mathbf{s}}(\mathbf{z})J^{a_k}(w_k)\dots J^{a_{q+1}}(w_{q+1})J^{a_{q-1}}(w_{q-1})\dots J^{a_1}(w_1)\rangle \\
&\quad -(K^a_b)_i \sum_{q=1}^{k}\oint_0 \frac{\mathrm{d}w_k}{2\pi i w_k}\cdots\oint_0 \frac{\mathrm{d}w_1}{2\pi i w_1}\oint_{w_q}\frac{\mathrm{d}z}{2\pi i(z-z_i)}\frac{i\varepsilon_{ba_qc}}{z-w_q}\langle \Phi_{\mathbf{s}}(\mathbf{z})J^{a_k}(w_k)\dots J^{c}(w_q)\dots J^{a_1}(w_1)\rangle \\
&= -(K^a_b)_i \sum_{j(\neq i)}\frac{t^b_j}{z_i-z_j}\oint_0 \frac{\mathrm{d}w_k}{2\pi i w_k}\cdots\oint_0 \frac{\mathrm{d}w_1}{2\pi i w_1}\langle \Phi_{\mathbf{s}}(\mathbf{z})J^{a_k}(w_k)\dots J^{a_1}(w_1)\rangle \\
&\quad +(K^a_{a_q})_i \sum_{q=1}^{k}\oint_0 \frac{\mathrm{d}w_k}{2\pi i w_k}\cdots\oint_0 \frac{\mathrm{d}w_1}{2\pi i w_1}\frac{1}{2(w_q-z_i)^2}\langle \Phi_{\mathbf{s}}(\mathbf{z})J^{a_k}(w_k)\dots J^{a_{q+1}}(w_{q+1})J^{a_{q-1}}(w_{q-1})\dots J^{a_1}(w_1)\rangle \\
&\quad -(K^a_b)_i \sum_{q=1}^{k}\oint_0 \frac{\mathrm{d}w_k}{2\pi i w_k}\cdots\oint_0 \frac{\mathrm{d}w_1}{2\pi i w_1}\frac{i\varepsilon_{ba_qc}}{w_q-z_i}\langle \Phi_{\mathbf{s}}(\mathbf{z})J^{a_k}(w_k)\dots J^{c}(w_q)\dots J^{a_1}(w_1)\rangle \\
&= -(K^a_b)_i \sum_{j(\neq i)}\frac{t^b_j}{z_i-z_j}\psi_{a_k\dots a_1} + \sum_{q=1}^{k}\frac{(K^a_{a_q})_i}{2z_i^2}\psi_{a_k\dots a_{q+1}a_{q-1}\dots a_1} \\
&\quad -\sum_{q=1}^{k}(K^a_b)_i i\varepsilon_{ba_qc}\oint_0 \frac{\mathrm{d}w_k}{2\pi i w_k}\cdots\oint_0 \frac{\mathrm{d}w_q}{2\pi i w_q}\frac{1}{w_q-z_i}\langle \Phi_{\mathbf{s}}(\mathbf{z})J^{a_k}(w_k)\dots J^{c}(w_q)(J^{a_{q-1}}_{-1}\dots J^{a_1}_{-1})(0)\rangle \\
&= -(K^a_b)_i \sum_{j(\neq i)}\frac{t^b_j}{z_i-z_j}\psi_{a_k\dots a_1} + \sum_{q=1}^{k}\frac{(K^a_{a_q})_i}{2z_i^2}\psi_{a_k\dots a_{q+1}a_{q-1}\dots a_1} - \sum_{q=1}^{k}\sum_{n=-1}^{q-1}(K^a_b)_i i\varepsilon_{ba_qc} \\
&\quad \times \oint_0 \frac{\mathrm{d}w_k}{2\pi i w_k}\cdots\oint_0 \frac{\mathrm{d}w_q}{2\pi i w_q}\frac{w_q^{-n-1}}{w_q-z_i}\langle \Phi_{\mathbf{s}}(\mathbf{z})J^{a_k}(w_k)\dots J^{a_{q+1}}(w_{q+1})(J^c_nJ^{a_{q-1}}_{-1}\dots J^{a_1}_{-1})(0)\rangle \\
&= -(K^a_b)_i \sum_{j(\neq i)}\frac{t^b_j}{z_i-z_j}\psi_{a_k\dots a_1} + \sum_{q=1}^{k}\frac{(K^a_{a_q})_i}{2z_i^2}\psi_{a_k\dots a_{q+1}a_{q-1}\dots a_1}
\end{aligned}
$$

$$+ (K_b^a)_i \sum_{q=1}^{k} \sum_{n=-1}^{q-1} \frac{i\varepsilon_{b a_q c}}{z_i^{n+2}} \langle \Phi_{\mathbf{s}}(\mathbf{z}) (J_{-1}^{a_k} \dots J_{-1}^{a_{q+1}} J_n^c J_{-1}^{a_{q-1}} \dots J_{-1}^{a_1})(0) \rangle \tag{C5}$$

Multiplying this equation by $2z_i$ gives

$$(K_b^a)_i \sum_{j(\neq i)} (w_{ij}+1) t_j^b \psi_{a_k \dots a_1} = \sum_{q=1}^{k} \frac{(K_{a_q}^a)_i}{z_i} \psi_{a_k \dots a_{q+1} a_{q-1} \dots a_1} + 2(K_b^a)_i \sum_{q=1}^{k} \sum_{n=-1}^{q-1} \frac{i\varepsilon_{b a_q c}}{z_i^{n+1}} \langle \Phi_{\mathbf{s}}(\mathbf{z}) (J_{-1}^{a_k} \dots J_{-1}^{a_{q+1}} J_n^c J_{-1}^{a_{q-1}} \dots J_{-1}^{a_1})(0) \rangle$$

The application of the spin operator T^a to $\psi_{a_k \dots a_1}$ can be written in terms of the Levi-Civita symbol,

$$\begin{aligned} T^b \psi_{a_k \dots a_1} &= \langle \phi_{s_1}(z_1) \dots \phi_{s_N}(z_N) (J_0^b J_{-1}^{a_k} \dots J_{-1}^{a_1})(0) \rangle \\ &= i \sum_{q=1}^{N} \varepsilon_{b a_q c} \psi_{a_k \dots a_{q+1} c a_{q-1} \dots q_1}. \end{aligned} \tag{C6}$$

Using this result, noting that $(K_b^a)_i t_i^b = 0$ and that the operator $\mathcal{C}_i^a$ can be written as $\mathcal{C}_i^a = (K_b^a)_i \sum_{j(\neq i)} w_{ij} t_j^b$, we get

$$\begin{aligned} &\mathcal{C}_i^a \psi_{a_k \dots a_1} \\ &= \sum_{q=1}^{k} \frac{(K_{a_q}^a)_i}{z_i} \psi_{a_k \dots a_{q+1} a_{q-1} \dots a_1} \\ &\quad + (K_b^a)_i T^b \psi_{a_k \dots a_1} \\ &\quad + 2(K_b^a)_i \sum_{q=2}^{k} \sum_{n=0}^{q-1} \frac{i\varepsilon_{b a_q c}}{z_i^{n+1}} \\ &\qquad \times \langle \Phi_{\mathbf{s}}(\mathbf{z}) (J_{-1}^{a_k} \dots J_{-1}^{a_{q+1}} J_n^c J_{-1}^{a_{q-1}} \dots J_{-1}^{a_1})(0) \rangle. \end{aligned}$$

Appendix D: Lattice Momentum of Wave Functions

Following Ref. 8, we define the lattice momentum operator as $p = -i \ln(\mathcal{T})$, where

$$\mathcal{T} = \mathcal{P}_{N,N-1} \mathcal{P}_{N-1,N-2} \dots \mathcal{P}_{2,1} \tag{D1}$$

is the translation operator and $\mathcal{P}_{ij} = 2 t_i^a t_j^a + 1/2$ is the operator that permutes the states of spin i and spin j. Note that the momentum is defined modulo 2π.

The wave function ψ_0 as given in Eq. (10) is equivalent to

$$\begin{aligned} &\tilde{\psi}_0(s_1, \dots, s_N) \\ &\quad = \delta_{\mathbf{s}} \prod_{p=1}^{N} e^{i\pi(p-1)(s_p+1)/2} \prod_{n<m} (z_n - z_m)^{(s_n s_m + 1)/2}, \end{aligned} \tag{D2}$$

because $\tilde{\psi}_0$ and ψ_0 differ only by a spin independent constant.

Applying $\mathcal{T}$ to $\tilde{\psi}_0$ gives

$$\begin{aligned} \mathcal{T} \tilde{\psi}_0 &= \delta_{\mathbf{s}} \prod_{p=1}^{N} e^{i\pi(p-1)(s_{p-1}+1)/2} \\ &\quad \times \prod_{n=1}^{N-1} \prod_{m=n+1}^{N} (z_n - z_m)^{(s_{n-1} s_{m-1} + 1)/2} \\ &= (-1)^{N/2} \tilde{\psi}_0 \end{aligned} \tag{D3}$$

Therefore the momentum of ψ_0 is

$$p_0 = \begin{cases} \pi & \text{for } N/2 \text{ odd} \\ 0 & \text{for } N/2 \text{ even} \end{cases} \tag{D4}$$

With $\mathcal{T} u_{-1}^a \mathcal{T}^{-1} = e^{-2\pi i/N} u_{-1}^a$, it follows that $\psi_{a_k \dots a_1}$ has the momentum $p_0 - 2\pi k/N$.

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